Problem III.4 ... size matters

8 points; průměr 4,20; řešilo 83 studentů

A sphere with a radius r rolls on a horizontal surface with a speed v. However, its path is blocked by a perpendicular step with the height h. Find the conditions under which the ball rolls onto the step and starts rolling along it without losing contact with the step. Under these conditions, determine its speed after it has crossed the step. Assume that all collisions are perfectly inelastic and the friction between the ball and the step is high. The step is angular and is oriented perpendicular to the direction of the sphere's motion.

Dodo had small wheels.

We start by finding conditions for the problem parameters under which the sphere overcomes the step. We can see right away that if $h \geq r$, then the ball, under the assumptions of the problem, will never climb the step. This is because the ball loses all of its horizontal velocity and all it has left is angular velocity. Which, on contact with the step, will cause the sphere to accelerate only in the vertical direction. So, even if the ball could be moved to a height greater than h, it would never reach the step because it would fall back in front of it.

Thus, we will only consider the cases where $h < r$. The staircase is square (we can imagine it as a long block), so the collision only brings the ball into contact with the top edge. The collision is perfectly rigid, so the sphere loses the normal component of velocity. By normal, we mean the line that passes through the edge of the obstacle and the center of the sphere, according to the figure 1.

Let us denote the normal's angle with the vertical direction as φ. Then, from the original velocity of the center of the sphere, v_0 only remains

$$v_0 \cos\varphi = v_0 \cdot \frac{r-h}{r} = v \cdot \left(1 - \frac{h}{r}\right) .$$

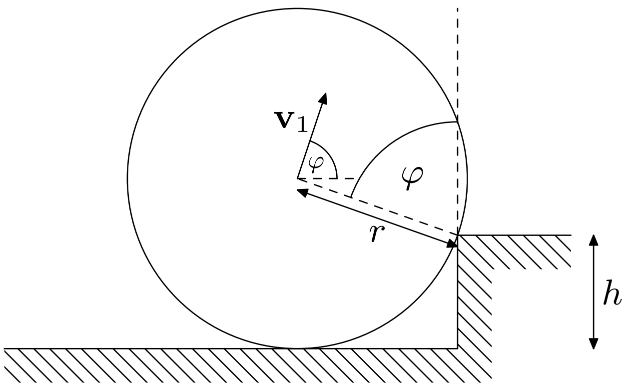

Fig. 1: Sketch of the situation.

This is not the ball's final velocity. It is just the velocity after the impact. This is because the ball will also collide with the step in the tangential direction which will affect its angular velocity (similar to how we spin the ball when we hit it with the table tennis bat). This collision is due to friction (which is large by problem statement). If the step was perfectly smooth, the rotation of the ball would not change.

Because the friction between the ball and the step is high (or, if you want, the collision is rigid), the point of contact between the ball and the step loses velocity immediately on impact. The point of contact has zero velocity if the following equation between the velocity of the sphere v_1 after the collision with the step and its angular velocity ω_1 holds

$$v_1 = \omega_1 r\,.$$

Now, we would like to calculate the final velocity v_1 of the center of the sphere after the impact. In a perpendicular impact, the sphere's velocity decreases from v_0 to $v_0 \cos\varphi$, and the angular velocity does not change. Then, the point of contact with the edge moves *against the direction* of the motion of the sphere. The friction force stops the motion of the point of contact, so it acts *in the direction* of the velocity $v_0 \cos\varphi$.

The total momentum is then increased by some Δp in the tangential direction of the collision and the angular momentum decreases by some $\Delta L = \Delta p r$ because the torque M from the frictional force F_t is equal to $F_\mathrm{t} r$.

We have then,

$$J\Delta\omega = mr\Delta v$$
$$\frac{2}{5} r\Delta\omega = \Delta v\,,$$

where $J = 2mr^2/5$ is moment of inertia of the sphere. The velocity of the sphere changed from $v_0 \cos\varphi$ to v_1 during the collision and the angular velocity from v_0/r to v_1/r. From this, we get

$$\frac{2}{5}(v_0 - v_1) = v_1 - v_0 \cos\varphi\,,$$

from which

$$v_1 = v_0 \cdot \left(\frac{2}{7} + \frac{5}{7}\cos\varphi\right) = v_0 \cdot \left(1 - \frac{5h}{7r}\right)\,.$$

From the velocity v_1, we can now determine if the ball will even cross the step. The limiting case occurs if the ball has zero velocity at the top. From the law of conservation of energy, we have

$$\frac{1}{2} m v_1^2 + \frac{1}{2} J \omega_1^2 > mgh\,,$$
$$v_0^2 > \frac{\frac{10}{7} gh}{\left(1 - \frac{5h}{7r}\right)^2}\,.$$

Under this condition, the sphere overcomes the step. Let us now consider that the initial velocity of the sphere satisfies this condition. Two cases can occur as the ball moves over the edge of the step.

First, the ball does not come off the step and merely rotates around its edge. The second possibility is that the ball's velocity is so great that as it rotates the centrifugal force overcomes the centripetal component of the gravitational force, and the ball loses contact with the step.

We will now investigate this possibility. The ball will never come off the edge if it is between the centrifugal and gravitational force, the inequality

$$m\frac{v^2}{r} < mg\cos\theta\,,$$

the angle θ is defined in the same way as φ, but for the situation when the ball no longer touches the horizontal pad it was on. The speed of the ball gradually decreases as it moves, and the angle θ decreases. Left side of the equation, therefore, decreases, and the right side increases. Therefore, the inequality is satisfied throughout the motion only when it is satisfied at the beginning, just after impact. If we plug the expression for v_1 into the inequality, we get

$$v_0^2 < \frac{g(r-h)}{\left(1-\frac{5h}{7r}\right)^2}\,.$$

So, we have upper and lower bounds for the initial velocity of the sphere v_0. At the same time, we see that if

$$\frac{\frac{10}{7}gh}{\left(1-\frac{5h}{7r}\right)^2} > \frac{g(r-h)}{\left(1-\frac{5h}{7r}\right)^2}\,,$$

$$h > \frac{7}{17}r\,,$$

the situation where the ball does not bounce at all cannot occur for any velocity v_0.

Consider now that r, h and v_0 satisfy the derived conditions and compute the velocity of the ball after it rolls onto the step. Since the point of contact of the ball with the edge of the step is not moving, there is no friction and energy is conserved as the ball rotates (as we considered when looking for the condition to overcome the step). For the velocity of the ball v_2 after rolling over the step, the following holds

$$\frac{1}{2}mv_2^2 + \frac{1}{2}J\omega_2^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}J\omega_1^2 - mgh\,.$$

Using the no-slip condition, we have $v_2 = \omega_2 r$, and thus

$$v_2 = \sqrt{v_1^2 - \frac{10}{7}gh} = \sqrt{v_0^2\left(1-\frac{5h}{7r}\right)^2 - \frac{10}{7}gh}\,.$$

As soon as the ball rolls and gets to the step, there are no more collisions, and therefore, the calculated v_2 is the velocity of its roll after it is over the step.

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