Problem I.5 ... cold water immersion in the summer

10 points; průměr 6,71; řešilo 109 studentů

In the winter, Matěj found a $0.5\,\mathrm{m}^3$ *bale of polystyrene and decided to use it. He made a cube-shaped box out of it. Then he cut the ice from a frozen pond, which he stored in the polystyrene cube in the cellar, where the temperature is constant* $9\,^\circ\mathrm{C}$*. How big should Matěj make the cube so that he has the largest amount of ice left in it after half a year? And how many kilograms of ice will he have left? Suppose that the ice from the pond has a temperature of exactly* $0\,^\circ\mathrm{C}$*. Ignore the volume of polystyrene used for the edges of the cube.*

Hint: The thermal conductivity coefficient is the easiest parameter of polystyrene to find.

Matěj borrowed a bundle of polystyrene from the building.

The bigger the box Matěj makes, the more ice it will be able to hold; however, by doing so, the walls get thinner, which will worsen insulation.

Assuming that Matěj built the cube watertight so no water would leak. The melting process of ice is relatively slow, so we will assume that during melting, there is ice and water inside the cube at a constant temperature of $0\,^\circ\mathrm{C}$. We can do this because water has a much higher thermal conductivity than polystyrene. So we work with a temperature difference of $\Delta T = 9\,\mathrm{K}$. At the same time, we will not take into consideration the effect of the air bubble that forms inside because ice decreases in volume as it melts.

Matěj has polystyrene with a volume of $V = 0.5\,\mathrm{m}^3$. If he makes a cube with edge a, the wall will be $d = V/S = V/6a^2$ thick, where $S = 6a^2$ is the surface of the cube and disregarding the material used for the vertices and edges of the cube.

Using the definitional relation for the thermal conductivity coefficient λ, we can calculate the energy that will leak through the polystyrene into the box in time $t = 0.5\,\mathrm{years} = 1.6\cdot 10^7\,\mathrm{s}$

$$\Delta E = \lambda \Delta T \frac{S}{d} t = \lambda \Delta T \frac{36a^4}{V} t\,,$$

where $\lambda = 0.035\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$ for polystyrene. Dividing this energy by the specific heat of fusion of ice $l_\mathrm{t} = 334\,000\,\mathrm{J{\cdot}kg^{-1}}$ will give us the mass of melted ice. Therefore, the mass of the remaining ice is

$$m = a^3\rho - \frac{\Delta E}{l_\mathrm{t}} = a^3\rho - \lambda \Delta T \frac{36a^4}{V l_\mathrm{t}} t\,,$$

where $\rho = 920\,\mathrm{kg{\cdot}m^{-3}}$ is a density of ice.

The maximum possible mass of the remaining ice is that for which the derivative of this expression is zero, that is

$$\frac{\mathrm{d}m}{\mathrm{d}a} = 0\,,$$

$$3a^2\rho - 4\lambda\Delta T \frac{36a^3}{V l_\mathrm{t}} t = 0\,,$$

$$\lambda \Delta T \frac{48a}{V l_\mathrm{t}} t = \rho\,,$$

$$a = \frac{\rho V l_\mathrm{t}}{48 \Delta T \lambda t} \doteq 0.64\,\mathrm{m}\,.$$

For a box with this side length, the walls have a thickness of $0.2\,\mathrm{m}$, so neglecting the material consumed on the vertices and edges of the cube is not very reasonable, but it will suffice for a rough estimate. After half a year, Matěj will have a whole $m = \left(\rho V l_\mathrm{t}/(48\Delta T\lambda t)\right)^3 \rho/4 = 59\,\mathrm{kg}$ of ice, which is still enough for regular summer body hardening.

In reality, however, there would be even more ice left because we did not consider the thermal conductivity of the water (or rather the air if the water was leaking out), which would help insulate the inner chunk of ice.

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