Problem I.E . . . dense ice

13 points; (chybí statistiky)

Measure the density of ice.

Karel's previous ice-problem was rejected, so he came up with another one.

Introduction

At first glance, it may seem that this is an effortless task. However, the ice melts quickly, which can be partially eliminated by cooling the instruments used for the measurements, but the experiment is still very demanding to perform accurately. We will present three methods by which ice density can be determined and evaluate their accuracy.

The density ϱ (mass) can be determined as the ratio of the mass m to the volume V

$$\varrho = \frac{m}{V} \,. \tag{1}$$

We can measure volume in several ways. The easiest is to measure the dimensions of the solid and calculate the volume. It is also possible to determine the volume by immersion in a liquid. However, since the ice has a lower density than water, you either need to tuck it under the surface or use a liquid with a lower density (such as oil).

Another possibility is to use Archimedes' law in the form

$$V = \frac{m}{\varrho_\mathrm{k}} \,.$$

Here we will need to add weight to the ice to make it sink and modify the above formula appropriately. The derivation of these relations can be found in the appendix at the end of this solution.

Methods of measurement

1st method – determining volume by cube dimensions The easiest way to determine the density of ice is to measure its mass and volume and then calculate it using the formula 1.

Two cylinder-shaped pieces of ice were used for the experiment. One was very small, about the size of a regular ice cube, and a metal container with a circular base with a diameter of 12 cm was used to make the other.

We measured the diameter of the trim piece by moving a ruler around the base and trying to place it in the center until we had measured the most significant distance (the diameter being the longest side of the circle). Conversely, we measured the height by trying to find the shortest (i.e., perpendicular) line joining the two figures. In this way, we significantly increased the accuracy of both measurements. A ruler was used for the measurements, and we estimated the measurement error to be 0.2 cm. Usually, the size of the smallest piece (or half of it) is taken as the error. In this case, however, we are measuring an uneven surface (the portion of ice we have created is not ideal). The height and diameter may differ at different places on the cylinder, so we estimated the measurement error to be slightly larger.

To measure the diameter of the larger cylinder, we needed to find the center of the base. We did this by drawing a line on the ice with a marker, finding its center, and drawing a perpendicular line through it. The intersection of the two axes thus formed was then taken as the center. The diameter of the base itself was measured so that the ruler's edge passed through three points, one of which was the center, and the other two points lay on the edge of the base.

This measurement was then made four times. The height of the cylinder was measured in the same manner as for the small ice, the measurement being made three times.

The weight was always determined using a kitchen scale to an accuracy of 1 g. This accuracy is very low; it would have been better to use a finer technical scale.

If we had chosen this first method of measurement, it would have been advisable to make considerably more measurements and determine the ice density as the average. In addition to repeating the experiment, it would be a good idea to choose a better shape of ice maker that would be easier to measure the dimensions, for example, a sizeable cube-shaped mold. The larger the ice, the smaller the relative error in calculating its dimensions.

2nd method – determination of volume by immersion in liquid Another way of measuring volume is to immerse an object in a liquid in a graduated cylinder, on the scale of which we read how much liquid the object has displaced and, therefore, what its volume is. To do this, a liquid of lower density than the body must completely submerge the object. In the case of ice, which has a density of about $0.917\,\mathrm{g{\cdot}cm^{-3}}$,[1] we cannot use water (with density $0.997\,\mathrm{g{\cdot}cm^{-3}}$). The lower density is, for example, that of rubbing alcohol or rapeseed oil, which we have used.

To make the results more accurate, we need to prevent some ice from melting. All instruments were, therefore, pre-cooled in a refrigerator. We also used chilled wooden tongs to handle the ice.

We poured chilled rapeseed oil into a graduated cylinder and read the volume from a scale. We used a kitchen scale to determine the weight of several pieces of ice and recorded everything in a table.

Since the ice cubes are relatively small, we used more of them per measurement (always a random number of about 4-5 cubes), which also reduced the relative error in measuring the weight or volume (the error was still 1 g, but instead of 9 g the ice weighed about 50 g). However, we did not determine a single cube's average weight or volume since the ice was formed so that the cubes differed slightly from one another.

After weighing, we carefully transferred the ice with tongs (to avoid significant heat transfer from our hands to the ice) into a graduated cylinder. If the cubes did not sink completely, we pushed them gently under the surface using a cooking pot. Since rapeseed oil is less dense than ice, the cubes did not float back up. Again, we used the scale on the graduated cylinder to determine the volume of its contents. We consistently subtracted from the lower (or always from the upper) meniscus so that the difference in these volumes corresponded to the importance of the ice placed. We tried to proceed very quickly as the ice melted and the importance changed.

The uncertainty of the volume measurement is determined as the value of the smallest piece, 5 mm. The samples were quite large, so we could distinguish even half a bit. Still, the resulting volume was obtained by the difference of these values, and the total error is then determined as the square root of the sum of the squares of the errors of the two measurements, $\sqrt{2.5^2 + 2.5^2} \doteq$ $\doteq 4$. Since a small error could have been introduced by misidentifying the position of the upper or lower end of the surface, we decided to leave the error larger - corresponding to the size of one slice.

[1] MIKULČÁK, Jiří. Mathematical, physical, and chemical tables for secondary schools. 14th ed. Prague: SPN, 1985. Auxiliary books for pupils (State Pedagogical Publishing House).

3rd method – determining density using Archimedes' law The last possible approach,[2] which we will present here, is to use Archimedes' law. As can be seen from the formula 9, we can use it to determine the volume of the submerged part of a body. Since ice has a lower density than water, it will float on the surface and never completely submerge. For this reason, when we created the ice, we added an object of a more significant, known density to the form (here, we used three steel nails). The smaller weight might fall off as the ice melts if we let it freeze at the ice's edge. Therefore, it is advisable to freeze only part of the water, place a metal object on the frozen surface, refill the ice container with water, and let it freeze again. This way, we can put the body roughly in the middle of the ice.

We then put the ice piece with nails into the water container and let it slowly melt. As the amount of ice diminished, the object slowly sank. We waited until such a part of the ice had melted that the cube was completely submerged and floating under the surface so that the buoyancy and gravitational forces were still in balance (i.e., the object was not yet falling to the bottom).

At this point, we pulled the object out and placed it in an empty container. Here we left it until all the ice had melted and then measured the water and nails mass.

Fig. 1: Ice

Results

1st method The small cylindrical ice cube used has a base diameter of (2.5 ± 0.2) cm and a height of (2.2 ± 0.2) cm.

Calculate the volume from the formula

$$V = \pi \left(\frac{d}{2}\right)^2 h\,.$$

The piece of ice used had a mass of $m = 9\,\mathrm{g}$. Now we can calculate the density from the formula 1.

As part of the measurement, we need to calculate the error of the result. If we know the errors of the measured quantities, we can use them and the formula for calculating the quantity A to

[2] Inspired by the text `http://fyzikalniolympiada.cz/archiv/58/fo58d1_r.pdf`

calculate its error σ_A. For simple formulas where all quantities occur only in the product or proportion, the relative error[3] of the quantity A is the square root of the sum of the squares of the relative errors of the measurands. If any of these quantities occurs to the n-th power, we take the relative error of that quantity n times

$$\sigma_A = A\sqrt{\sum \left(\frac{n\sigma_{x_i}}{x_i}\right)^2}\,. \qquad (2)$$

Here, n represents the power with which this quantity occurs in the formula.

In this case, the relation for calculating the error is of the form.

$$\sigma_\varrho = \varrho\sqrt{\left(\frac{\sigma_h}{h}\right)^2 + \left(\frac{2\sigma_d}{d}\right)^2 + \left(\frac{\sigma_m}{m}\right)^2}\,.$$

Using this procedure, we determined the density of ice as $\varrho_{\text{led1}} = (0.8 \pm 0.3)\ \text{g}\cdot\text{cm}^{-3}$. We rounded the error to one valid digit, then rounded the result so that its last digit was of the same order as the error.

For the large cylinder, the following values were measured (listed in the table with the error obtained as the square root of the sum of the square of the standard deviation and the gauge error).

Tab. 1: Dimensions of ice of mass $m = 354\,\text{g}$ in the shape of a large cylinder.

č.	$\frac{d}{\text{cm}}$	$\frac{h}{\text{cm}}$
1	12.5	3.0
2	12.6	3.1
3	12.8	2.7
4	–	2.9
diameter	12.6 ± 0.1	2.9 ± 0.2

The volume was again calculated from the formula

$$V = \pi \left(\frac{d}{2}\right)^2 h\,.$$

And density as the ratio of mass to volume. The error was again determined as

$$\sigma_\varrho = \varrho\sqrt{\left(\frac{\sigma_h}{h}\right)^2 + \left(\frac{2\sigma_d}{d}\right)^2 + \left(\frac{\sigma_m}{m}\right)^2}\,.$$

The resulting density value is then $\varrho_{\text{led1large}} = (0.94 \pm 0.06)\ \text{g}\cdot\text{cm}^{-3}$.

Tab. 2: Experimentally determined ice density values. The volume of ice was measured by immersion in rapeseed oil.

No	$\frac{m}{\mathrm{g}}$	$\frac{V_{\text{bez ledu}}}{\mathrm{cm}^3}$	$\frac{V_{\text{s ledem}}}{\mathrm{cm}^3}$	$\frac{V_{\text{led}}}{\mathrm{cm}^3}$	$\frac{\varrho}{\mathrm{g\cdot cm^{-3}}}$
1	32	150	185	35	$0,9 \pm 0,1$
2	42	235	280	45	$0,9 \pm 0,1$
3	44	437	485	48	$0,9 \pm 0,1$
4	46	350	400	50	$0,92 \pm 0,09$
5	51	415	470	55	$0,93 \pm 0,09$
6	43	420	465	45	$1,0 \pm 0,1$

2nd method The measurements with rapeseed oil were performed 6 times in total, and the measured values are shown in table 2.

We calculate the error as

$$\sigma_\varrho = \varrho\sqrt{\left(\frac{\sigma_m}{m}\right)^2 + \left(\frac{\sigma_V}{V}\right)^2}\,. \tag{3}$$

We determine the density as the average of the values from the individual measurements, where we do not include the last measured (sixth) value since it deviates from the others (the value falls outside the 3sigma interval). This measurement probably has some significant error, which would unnecessarily bias our result. The mean value should be supplemented with a standard deviation, i.e., an indication of how far the measured values differ from the mean. The formula defines this quantity.

$$\sigma_{\text{sm.odch.}} = \sqrt{\sum \frac{(\bar{x} - x_i)^2}{n\,(n-1)}}\,.$$

We must not forget the error of the method, which we have listed as the error of the individual densities in the previous table. Since we have always made measurements for different amounts of ice, the method error is reduced $\sqrt{N}$ times to $0.1\,\mathrm{g\cdot cm^{-3}}/\sqrt{6} = 0.04\,\mathrm{g\cdot cm^{-3}}$ (the values are not always the same). The systematic error of the average value is smaller since the systematic errors caused by, for example, the cube always weighing 7.4 g and the scale showing an inaccurate value of 8 g (same with volume) cancel out when averaging. There were probably cases where the weight (or volume) was rounded up.

So the total error is $\sigma_c = \sqrt{\sigma^2_{\text{sm.odch}} + \sigma^2_m} = \sqrt{0.04^2 + 0.004^2} = 0.04\,\mathrm{g\cdot cm^{-3}}$. We could also notice in the calculation that the standard deviation is negligible compared to the error of the method.

The density thus determined has the value $\varrho_{\text{led2}} = (0.92 \pm 0.04)\ \mathrm{g\cdot cm^{-3}}$.

3. method The experiment with a weight in a piece of ice was performed twice, each time with three steel nails. We have reported the data in table 3, including the results calculated

[3] Relative error is the ratio of the absolute error of the measurand to its value.

from the formula 10. We substituted densities from the tables $\varrho_\mathrm{v} = \varrho_\mathrm{water} = 0.997\,\mathrm{g\cdot cm^{-3}}$ and $\varrho_\mathrm{z} = \varrho_\mathrm{ocel} = 7.850\,\mathrm{g\cdot cm^{-3}}$. We determined the error using the error transfer formula 2 as

$$\sigma = \varrho\sqrt{\left(\frac{\sigma_{m_\mathrm{v}}}{m_\mathrm{v}}\right)^2 + \left(\frac{\sigma_{m_\mathrm{z}}}{m_\mathrm{z}}\right)^2}\,.$$

Tab. 3: Experimentally determined values of ice density using Archimedes' law.

No	$\frac{m_\mathrm{z}}{\mathrm{g}}$	$\frac{m_\mathrm{v}}{\mathrm{g}}$	$\frac{\varrho}{\mathrm{g\cdot cm^{-3}}}$	$\frac{\varrho_{schybou}}{\mathrm{g\cdot cm^{-3}}}$
1	14	109	0,899 2	$0,90 \pm 0,06$
2	14	106	0,896 6	$0,90 \pm 0,06$

For a more accurate measurement, we would still need to measure the temperature of the water bath to get the correct value for its density (likewise, we would need to recalculate the density of steel for the temperature 0 °C that the ice had. In our case, however, the error due to the density at 25 °C is negligible compared to the error in determining the mass.

Discussion

The first method is wildly inaccurate for a smaller piece of ice, and we determined the density as $\varrho_\mathrm{led1} = (0.8 \pm 0.3)\ \mathrm{g\cdot cm^{-3}}$. Thus, within the margin of error, our determined density may be as large as, or even more significant than, the density of water. If we were to measure the density of ice by this method, we would need to make many more measurements and use better-measuring instruments. We can also significantly improve by using a larger piece of ice. For a cylinder with more than 4 times the diameter of the base, we got a value of $\varrho_\mathrm{led1large} = (0.94 \pm 0.06)\ \mathrm{g\cdot cm^{-3}}$, so our error was reduced by order of magnitude.

The density of the ice determined by the oil immersion method has a value of $\varrho_\mathrm{led2} = (0.922 \pm 0.004)\ \mathrm{g\cdot cm^{-3}}$. Within the error, our measured ice density almost coincides with the tabulated value of[4] $\varrho_\mathrm{tab.} = 0.917\,\mathrm{g/cm^3}$.

In the second method, we obtain a reasonably accurate value (unlike the first method). One way to increase the accuracy would be to use engineering scales that would give us more valid digits. However, when we look more closely at the formula for calculating the error and try to plug in error in the mass measurement 0 g, we see that the overall error is almost unchanged. Therefore, to reduce the general error, we need to refine the graduated cylinder scale or use a different method of measuring density (see the third method).

The deviations from the actual value may have been due to some systematic errors that are not canceled out by repeated measurements - these are errors caused mainly by the rapid thawing of the ice. When measuring the weight, the ice thaws, and the tongs pick up a cube slightly lighter than its original weight. This should increase the measured density (we then put a cube with a smaller volume into the oil). The ice then melts even in water. Since ice is less dense than water, when it partially thaws, the volume decreases, which will also increase the measured value. A slight reduction in volume also occurs because some of the oil sticks to the rod when the ice cubes sink. These measurement errors may be one of the reasons why

[4] MIKULČÁK, Jiří. Mathematical, physical, and chemical tables for secondary schools. 14th ed. Prague: SPN, 1985. Auxiliary books for pupils (State Pedagogical Publishing House).

our measured value is larger than the tabulated value in almost all measurements. However, if we proceed quickly and have all instruments and oil subcooled to the lowest possible temperature, these differences should be almost negligible compared to the inaccuracy of the gauges. Moreover, when ice cubes are formed, tiny bubbles enter the cubes, reducing the overall density so that the resulting deviation from the actual value is not too significant. A systematic error probably contributed to the slight deviation from the tabulated value (even within the error). Otherwise, the measurement is relatively accurate, and the result corresponds reasonably well to the expected (tabulated) value.

A third method, which does not measure volume and uses Archimedes' law to determine density, might be the best. Here we are only measuring mass. The accuracy of this measurement can easily be improved by using engineering scales. The disadvantage of this method is the need for a weight of known density (if we were to verify it experimentally at home using, for example, the second method, we would again need to measure the volume, which would reduce our accuracy). With this method, we have determined the density as $\varrho_{\text{led3}} = (0.90 \pm 0.06)\ \text{g}{\cdot}\text{cm}^{-3}$. Both measurements may seem to yield a density value significantly less than that given in the tables. Still, the result is burdened with an error that makes it impossible to determine the exact value. The possible lower density may have been due to the more significant number of bubbles in the ice formed, which reduced the resulting density somewhat. The increased bubble content may be due, among other things, to the presence of nails that interfere with the ice's natural freezing. This would be suggested by the observation that we did not observe such a large number of bubbles in the second method, where the ice froze independently.

If better scales were used, the measurement error would be comparable to the standard deviation of the second method.

The third method, although more accurate in measuring instruments than the second, can be more demanding in terms of speed and experimenter error - the ice needs to be pulled out at the right time.

Closure

The first method of determining ice volume using measured dimensions was very inaccurate. The result $\varrho_{\text{led1}} = (0.8 \pm 0.3)\ \text{g}{\cdot}\text{cm}^{-3}$ was burdened with a large error. For a cylinder many times larger, however, we obtained a good result $\varrho_{\text{led1large}} = (0.94 \pm 0.06)\ \text{g}{\cdot}\text{cm}^{-3}$. The inaccuracy of the volume measurement was improved in the second method, with the error reduced by order of magnitude when measuring the volume in the graduated cylinder. The resulting density $\varrho_{\text{led2}} = (0.92 \pm 0.04)\ \text{g}{\cdot}\text{cm}^{-3}$ within the error nearly matches the tabulated value $\varrho_{\text{led-table}} = 0.917\,\text{g}{\cdot}\text{cm}^{-3}$. The third method determined the ice density as $\varrho_{led3} = (0.90 \pm 0.06)\ \text{g}{\cdot}\text{cm}^{-3}$.

Addendum

To determine the density of a body, Archimedes' law can be used, which we present here in its familiar form: a body immersed in a liquid is supercharged by a force whose magnitude is equal to the gravity of the liquid displaced by the body. We shall now derive this statement, and by calculation, we shall obtain the formula we shall use for the experiment. On a body of mass m in the homogeneous gravitational field of the Earth with a gravitational acceleration g, there is a gravitational force which we determine from the formula.

$$F_{\text{g}} = mg\,. \tag{4}$$

However, if we immerse this body in a liquid, it will also be subject to a buoyant force. The molecules of the liquid are also in the Earth's gravitational field, and a gravitational force acts on a layer of liquid of density ϱ_k at a distance h from the surface.

$$F_\mathrm{g} = mg = \varrho_k S\ hg\,, \tag{5}$$

Where S is the size of the cross-sectional area of this column of liquid. The fluid, therefore, exerts pressure on the layer we are examining.

$$p_\mathrm{g} = mg = \varrho_k hg\,. \tag{6}$$

Let us now imaginatively cut the body immersed in this liquid into small cylinders of cross-section dS perpendicular to the surface. On the upper base of the i-th cylinder, which is at a height h_1, the fluid exerts a force F_1, while on the lower base at a depth h_2, it exerts a force F_2. The resultant buoyant force on this cylinder is obtained as the difference between these two forces (geometrically, it is a sum since the forces act in opposition to each other - the liquid exerts pressure on the upper platform in the direction of the bottom, while the lower platform exerts a force that tends to lift the body). We, therefore, determine the force on such a small cylinder as

$$F_{\mathrm{vz}_i} = F_2 - F_1 = \varrho_k g\,\mathrm{d}Sh_2 - \varrho_k g\,\mathrm{d}Sh_1 = \varrho_k g\,\mathrm{d}S(h_2 - h_1)\,. \tag{7}$$

The total lift force on the roller is then obtained by summing the forces on the individual rollers. If we have made the rollers infinitesimally small, the sum goes into the integral, but it is still intuitive to view the integral as the sum of

$$F_\mathrm{vz} = \int \varrho_k g \Delta h\,\mathrm{d}S = \varrho_k g \int \Delta h\,\mathrm{d}S = \varrho_k g V'\,, \tag{8}$$

V' is the volume of the body's part immersed in the liquid.

The body remains at rest (or in uniform rectilinear motion) if the resultant of all forces acting on it is zero. Therefore, the body will become stationary by equalizing the gravitational and buoyant forces.

$$\begin{aligned} F_\mathrm{vz} &= F_g \\ \varrho_\mathrm{k} g V' &= mg \end{aligned} \tag{9}$$

We see that we can only determine the volume of the submerged part of the body in this way. If we were to immerse the ice in water, it would only be partially submerged due to its lower density. For complete immersion, we need to add weight to the ice, increasing the total density of the resulting body. We now show how the relation 9 we derived changes.

If the weight in the ice is light enough, the body will not sink completely. It will sink below the surface only after some ice has melted. Let us focus on this moment when the body is fully submerged, but the forces are still in equilibrium (the body does not sink but floats). At this point, the total volume of the body can be determined from our formula 9.

$$V = \frac{m}{\varrho_\mathrm{k}}\,,$$

Which we can modify using the fact that the mass m is the sum of the mass of the weights m_z and the water/ice m_l, V_v is the volume of the water, V_z is the volume of the weights, and ϱ_z is the density of the weights. The formula then takes the form by substituting the density of the liquid ϱ_k for the density of the water ϱ_v.

$$\varrho_\mathrm{led} = \frac{m_\mathrm{l}}{V_\mathrm{l}} = \frac{m_\mathrm{l}}{V - V_\mathrm{z}} = \frac{m_\mathrm{l}}{\frac{m}{\varrho_\mathrm{v}} - V_\mathrm{z}} = \frac{m_\mathrm{l}}{\frac{m}{\varrho_\mathrm{v}} - V_\mathrm{z}} \frac{\frac{1}{V_\mathrm{v}}}{\frac{1}{V_\mathrm{v}}} = \frac{\varrho_\mathrm{v}}{1 + \frac{m_\mathrm{z}\varrho_\mathrm{v}}{m_\mathrm{v}}\left(\frac{1}{\varrho_\mathrm{v}} - \frac{1}{\varrho_\mathrm{z}}\right)} \,. \tag{10}$$

Kateřina Charvátová
katerina.charvatova@fykos.org

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